The King's School Canterbury

13+ Scholarship Examinations 2018

MATHEMATICS I

75 minutes (plus 5 minutes reading time)

Use the reading time wisely; gain an overview of the paper and start to think how you will answer the questions.

Do as many questions as you can (clearly numbered) on the lined paper provided. Clearly name each sheet used with your full name. You do not have to attempt the questions in numerical order.

The questions are not of equal length nor mark allocation. Move on quickly if stuck; you are not expected to finish all the questions.

*You are expected to use a calculator where appropriate, but you must show **full and clear working**, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments. You can sometimes, however, manage to complete later parts of questions, even if you have failed to answer the earlier sections.

This paper has fifteen questions.

1 Donald says Vladimir tells lies.

Vladimir says Donald tells lies.

Teresa says Donald and Vladimir both tell lies.

Who is telling the truth?

2 From where I am standing, two adjacent corners of a square are respectively 13 m east of me and 43 m north of me.

What is the **area** of the square?

3 (a) Explain carefully why 1 km^2 = 1 000 000 m^2 .

Contrary to the popular view that this island is rather crowded, only about 1% of all UK land is covered by people's homes.

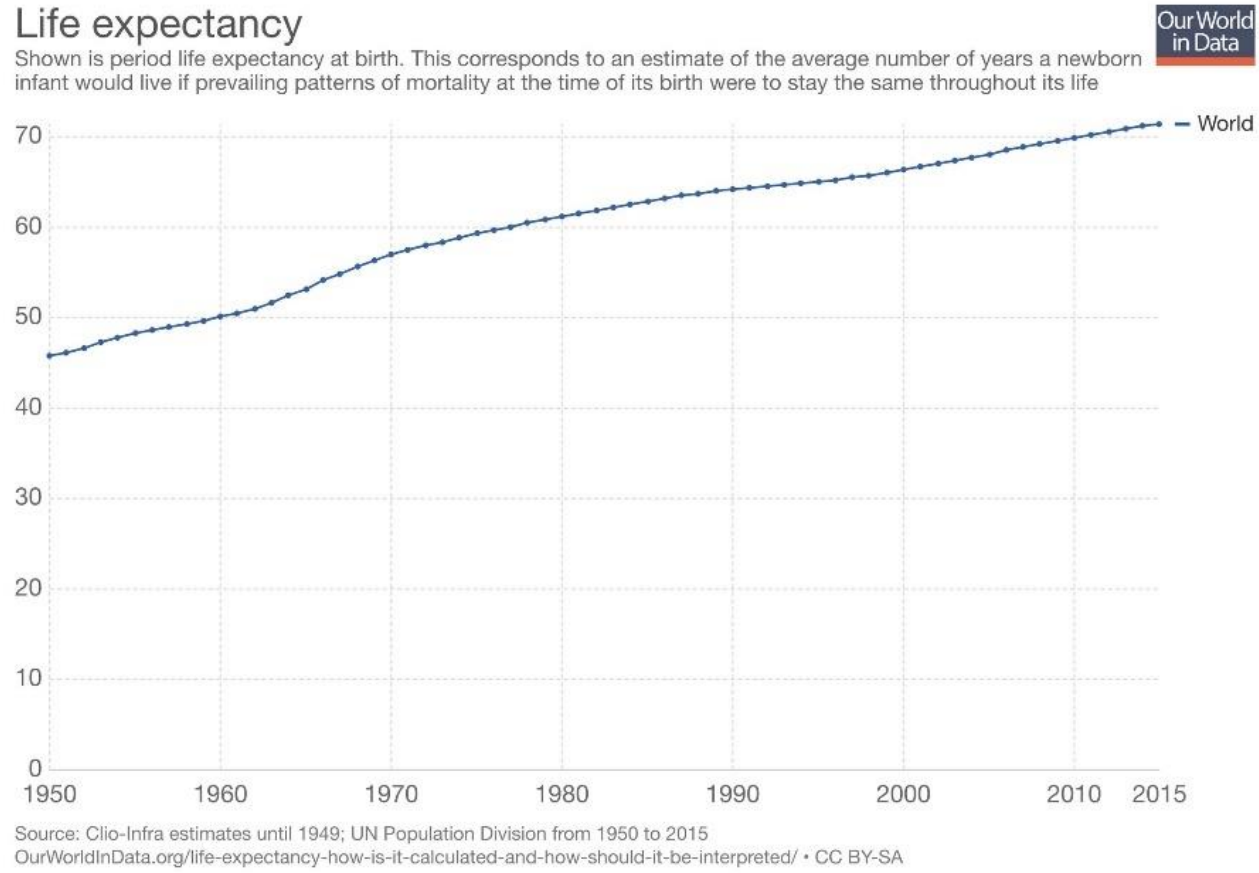
(b) Considering England only and using the data from 2011 given below, what is the mean average area inhabited by each person in England (i.e. based on 1% of the total land area being inhabited)? Give your answer in appropriate units.

- England land area: 130279 km^2
- England population: 52697866

The Surrey district of Woking (land area 63.57 km^2) has an astonishing 11% of its land area taken up by golf courses.

(c) Using your answer to (b), how many extra people could we settle on the golf courses of Woking?

4



From the graph above, global average life expectancy increased from 45 years in 1950 to 71 years in 2015.

A says: "this means that on average the world population has gained 9.5 hours of expected lifetime every single day".

B says: "so, the world population did not get 24 hours closer to death every day, but 14.5 hours"

(a) Show calculations to support these statements.

(b) Do you think these statements make sense? Make a couple of brief comments.

5 Before metric measure become more widespread in its use there were various scales used for weights, including **avoirdupois** (used for everyday items) and **troy** (used for precious metals and gemstones).

In the Imperial Weights and Measures Act of 1824, the avoirdupois pound was defined as 7000 grains exactly.

Note: one grain is identical in both troy and avoirdupois measures.

Avoirdupois		**Troy**	
16 drams	1 ounce	24 grains	1 dwt (pennyweight)
16 ounces	1 pound	20 dwt	1 ounce
		12 ounces	1 pound

A says: “the troy ounce is bigger than the avoirdupois ounce”.

B says: “that’s crazy, because the troy pound is smaller than the avoirdupois pound!”

Making your calculations clear, show that both these statements are correct.

6 In another scholarship examination (not this one) a candidate sits a forty-question test, and she gets *c* questions correct, *w* questions wrong and leaves *b* questions blank.

The scoring system is;

- Correct answer: 5 points
- Wrong answer: -1 points
- Blank answer: 0 points

Write down an expression for the candidate’s total score in the test.

7 (a) Explain why increasing a quantity by 1% may be calculated by multiplying the original by 1.01.

Alice, Ben and Cuthbert decide to spend an entire year (365 days) preparing for the scholarship examination.

They each start with a strength “rating” score of 1000. Consider this to be day zero.

Alice decides to try 1% harder every day (including day 1) which means her rating will rise by 1% from whatever it was the day before.

(b) What is her rating after three days? Give your answer to three decimal places.

(c) Explain why the calculation below will give Alice’s rating after one year, and work it out on your calculator.

$$1000 \times 1.01^{365}$$

Ben just maintains his original effort so his rating stays at 1000 throughout the year.

Cuthbert, however, from day zero, decides immediately to reduce his effort and tries 1% less hard every day for the whole year.

(d) What is the equivalent multiplier to (a) for finding a 1% decrease?

(e) Now write down a calculation (similar to in part (c)) to give Cuthbert's rating after 365 days, and work it out on your calculator.

(f) Write a couple of brief comments to compare the ratings for Alice, Ben and Cuthbert as they reach the end of their year of preparation.

8 In this question you may well need the standard form button on your calculator, and perhaps the negative sign-change button too (often labelled **(-)**).

Reminder: standard index form is a number written as $A \times 10^n$

with n a whole number (positive or negative), and A is such that $1 \leq A < 10$.

Professor Stephen Hawking (1942-2018) once said that one of his best ideas was a formula to calculate a quantity called **entropy** in a black hole in space. You do not need to know what entropy is (or anything else about black holes, for that matter) to answer this question. Here is the formula.

$$S = \frac{\pi A k c^3}{2hG}.$$

If the black hole has surface area $A = 2 \times 10^{10}$ m^2 use the data below to calculate the corresponding entropy S. Write down all the figures on your calculator display, giving your answer in standard index form. Do not worry about the units of your answer.

k	$1.38064852 \times 10^{-23}$ J/K	Boltzmann's constant
h	$6.62607004 \times 10^{-34}$ J s	Planck constant
G	6.67408×10^{-11} m^3 kg^{-1} s^{-2}	Newton's gravitational constant
c	299 792 458 ms^{-1}	Speed of light

9 In another scholarship examination (not this one), nine candidates must sit four mathematics papers over four days. They sit each of these examinations in a square array of 3×3 desks.

(a) Explain why there are 84 essentially different ways of choosing three candidates to sit in the front row (in any order) on day 1.

In fact, the main seating rule is that each candidate must sit on the same row as any other candidate **only once** across the four days. So, in the starting arrangement below, none of 1,2,3 (etc.) is allowed to sit in the same row as each other for the following three days.

123
456
789

(b) Write out the essentially unique solution to the seating plan for the following three days.

10 Consider the counting numbers from one to one billion.

Let's start by pairing off the numbers from each end of the sequence (omitting, for a moment, the billion itself and counting zero as the first term in the series). We write the numbers with the redundant zeroes included (where appropriate) so all the numbers below have nine digits.

000 000 000	999 999 999
000 000 001	999 999 998
000 000 002	999 999 997
000 000 003	999 999 996
---	---
135 286 504	864 713 495
---	---
327 456 182	672 543 817
---	---
499 999 998	500 000 001
499 999 999	500 000 000

(a) How many pairs of numbers are there in the above list?

(b) What is the sum of the digits of any pair of the numbers in the above list?

(c) Including the one billion missing in the above, what is the sum of all the digits used in writing down the numbers from one to a billion?

11 (a) Last week I drove 100 miles in 90 minutes. My average speed during the first 30 minutes was 75 mph, and my average speed during the second 30 minutes was 68 mph. What was my average speed, in mph, during the last 30 minutes?

(b) (i) Another time I drive on a motorway with an average speed restriction of 50 mph across a ten-mile stretch. After nine miles I realise I have been driving at average speed 65 mph, so I need to slow down for the last mile to avoid a speeding ticket. Assuming my deceleration is instantaneous, at what speed must I drive for the last mile so that my mean average speed across the ten miles is exactly 50 mph?

(ii) Comment on how realistic you consider your answer to (b)(i) to be.

12 The table given below shows the number of candidates who fail another scholarship examination in three main subjects: mathematics, English and Science. It also shows who failed more than one paper. For instance, eight candidates failed Science and, of those, six failed both mathematics and Science and, of those, two failed all three.

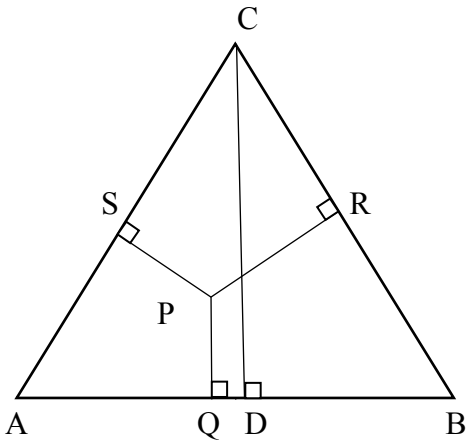
Paper(s)	M	E	S	M&E	M&S	E&S	M&E&S
Failures	7	6	8	3	6	3	2

There are 27 candidates for this examination. If I pick one student at random, what is the probability that this candidate has **passed** all three papers?

13 Consider an equilateral triangle ΔABC with each side of length 2m.

(a) What is the height of triangle ΔABC (CD in the diagram below)?

(b) What is the area of triangle ΔABC?



Professor John Roe (1959-2018) claimed: pick any point P inside the equilateral triangle and then the sum total of the perpendicular distances from the three sides will always be the same.

(c) What are the areas of triangles ΔAPB, ΔBPC and ΔCPA ?

(d) By comparing these answers with your answer to (b), show that Professor Roe was correct.

14 Solve the simultaneous equations

$$\frac{4}{x}+\frac{1}{y}=\frac{29}{4}$$

$$\frac{5}{x}-\frac{2}{y}=5$$

15 In this question we consider a historical problem from Fibonacci's book *Liber Abaci* (1202):

If a lion eats a sheep in four hours, a leopard eats a sheep in five hours, and a bear eats a sheep in six hours, how long would they all take to eat one sheep together?

Show your working carefully and give your answer (in hours) as an exact fraction.

[Assume they do not get in each other's way.]

END OF QUESTION PAPER