The King's School Canterbury

13+ Scholarship Examinations 2018

MATHEMATICS II

1 hour (including five minutes suggested reading time)

Use the reading time wisely; gain an overview of the paper and start to think of how you will answer the questions.

Do as many questions as you can (clearly numbered) on the lined paper provided. Clearly name each sheet used.

The questions are not of equal length or mark allocation. Move on quickly if stuck. ***You are not expected to finish everything.***

You are expected to use a calculator where appropriate, but you must show ***full and clear working****, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments.

This paper has ten questions. You can attempt them in any order.

1 In another scholarship examination, candidates sit six papers in total, in six time slots all on one day! Of these, they must take three different mathematics papers: number, algebra and shape.

If the rule is that we are not allowed to sit two mathematics papers in **consecutive** slots, in how many different ways can we arrange the mathematics papers? Note that we are not interested in what is being taken in the other slots, but (say) arrangement N*A*S* is different to N*A**S.

2 This question is taken from the University of London Elementary Mathematics examination in 1940.

A garrison of 1075 men could exist on full rations for 30 days.

After 16 days on full rations the garrison is augmented by 129 men and, at the same time, the stock of the existing provisions is increased by 60% by means of parachute supplies.

How much longer can the augmented garrison hold out on *half* rations?

See if you can answer the question, showing all your working. [Vocab: augmented means added to.]

3 Post Brexit (the UK leaving the European Union) it is decided that the country will revert to the pre-decimal currency system, of **pounds**, **shillings** and **pence** (abbreviated £ s d), which ran from before the Norman Conquest until 1971.

There were twelve pence in a shilling and twenty shillings in a pound. The half-penny is also reintroduced.

Mr Stuart buys a Royal Wedding commemorative plate and later finds that its value has gone up by 35%.

It is now worth £69 18s 7 ½ d.

What did he pay for it originally? Give your answer in £ s d.

4 Four players play a two-player dice game where they each have their own, non-standard six-sided die. The two players roll their dice and the higher score wins.

The dice are numbered like this:

A	1	1	5	5	5	5

B	4	4	4	4	4	4

C	3	3	3	3	7	7

D	2	2	2	6	6	6

Here is a table for A versus B, showing who wins in each cell

AvB	4	4	4	4	4	4
1	B	B	B	B	B	B
1	B	B	B	B	B	B
5	A	A	A	A	A	A
5	A	A	A	A	A	A
5	A	A	A	A	A	A
5	A	A	A	A	A	A

(a) Clearly A wins more often; what is the probability that A wins each game against B?

(b) Without spending time drawing grid-lines etc., quickly draw similar tables for B versus C, C versus D, and D versus A. In each case describe which player's die is stronger and what the probability is of winning for the stronger player.

(c) Write some brief comments about which is the strongest die here, or anything else relevant you wish to mention.

5 Pansy has been given a new game called Dobble (also known as Spot-It!)

There are 57 game cards each containing eight pictures of various symbols. There are 57 different symbols within the full set of cards.

One way to play the game is like an extended version of “Snap”. Players turn over two cards and then everyone must try to be first to spot the **unique** matching symbol on each pair of cards (and shout it out). Note: the cards are designed so that there is always **exactly one** common symbol between any pair in the pack.

The winning player each time then keeps those cards and the person with the most cards at the end is the final winner.

(a) Why must each symbol appear eight times in total in the pack of cards?

Pansy looks at the 57 cards and removes the eight cards which contain a dinosaur symbol.

(b) Explain carefully why all 57 different symbols appear on these eight cards.

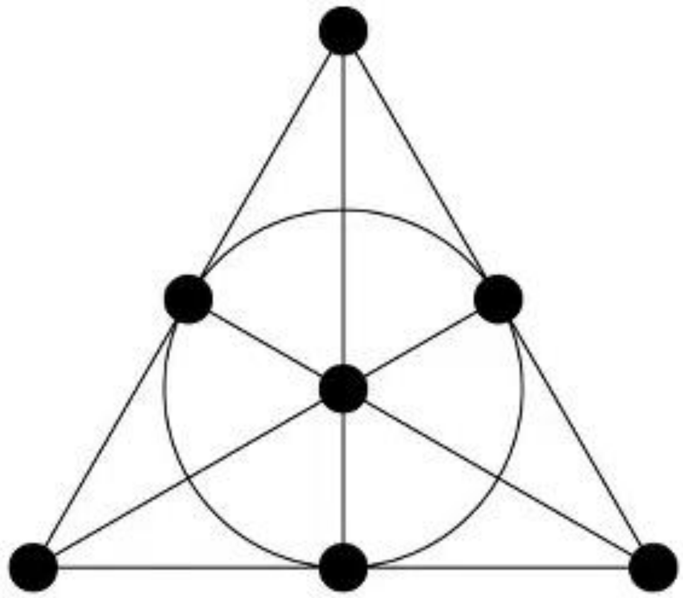
Trying to understand it better she now considers a simpler version of the game, this time with **seven** cards, and **seven** different symbols, with **three** symbols on each card. Once again, any pair of cards must have only **one** matching symbol.

The symbols on the cards are: Anchor, Bomb, Cheese, Dinosaur, Exclamation Mark, Fish, Ghost (abbreviated A, B, C, D, E, F, G).

(c) How many times will each symbol appear in this pack of cards?

(d) Write out the complete set of seven cards (use the abbreviations).

(e) What is the connection between this set of cards and the diagram opposite? There are seven lines (one of them being a circle), and there are seven points. Note that each line passes through three points, and any two points only have one line between them.



6 In an extraordinary development, someone has cheated during this examination. After the paper, Mr Miles questions the four suspects.

Walter says: Xavier cheated.
Xavier says: Walter is lying.
Yannick says: Walter cheated.
Zebedee says: I did not cheat!

(a) Suppose exactly one of these statements is true. Who cheated?

(b) Suppose exactly one of these statements is false. Who cheated?

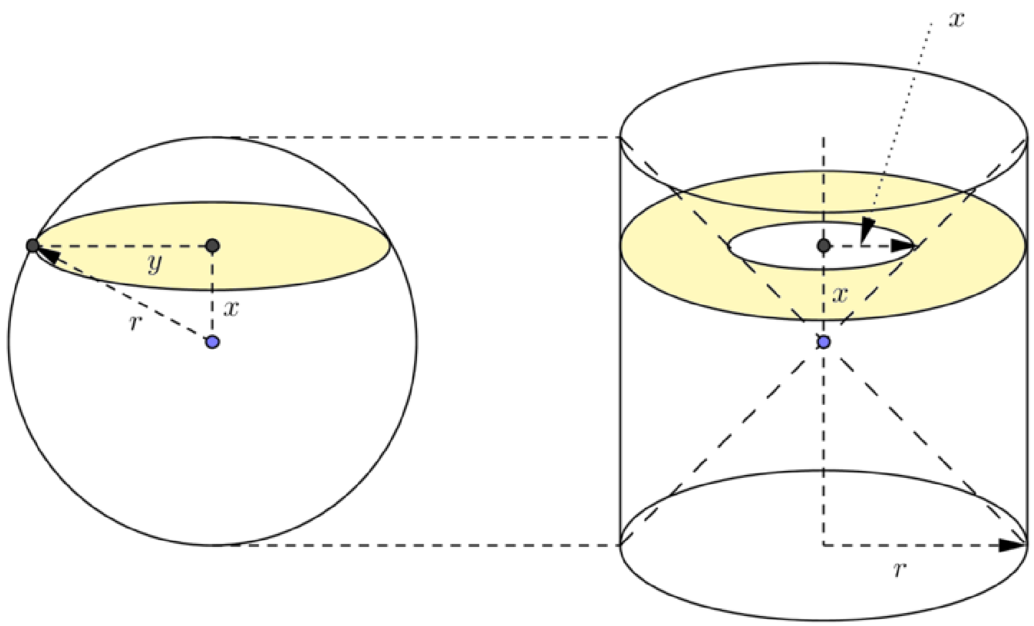
7 Mr Felton and Mr Mawby live at each end of a road. They each run from their own house to the other's and back. They each run at their own constant speed. They pass each other the first time a distance x from Mr Felton's house, then reach each other's house and immediately return to their own. On the way back they meet a second time a distance y from Mr Mawby's house.

Find an expression for the distance between their houses in terms of x and y.

8 In this question we will use Cavalieri's principle, which states that if corresponding cross-sectional areas of two figures are equal for every such "slice", then the two figures must have the same volume.

We seek a formula for the volume of a sphere, and we will assume two other volume formulae to be known already:

- For a cylinder of radius r and height h we have $V = \pi r^2 h$
- For a cone of radius r and height h we have $V = \frac{1}{3}\pi r^2 h$



In the diagrams we have a sphere on the left and, on the right, a cylinder with two cones cut out (which meet in the point shown in the very centre of the cylinder). Note that the sphere of diameter $2r$ would fit perfectly inside the cylinder of height $2r$.

(a) Explain why the shaded circle in the sphere has area $A = \pi(r^2 - x^2)$.

(b) Explain why the shaded annulus (ring) in the partly hollowed-out cylinder has area $A = \pi r^2 - \pi x^2$.

(c) What does this mean we can say about the two shaded areas?

(d) What can we deduce from this using Cavalieri's principle above?

(e) Explain why the volume of the cylinder is $V = 2\pi r^3$.

(f) Explain why the volume of the two cones is $V = \frac{2}{3}\pi r^3$.

(g) Use (e) and (f) to find the formula for the volume of the partly hollowed-out cylinder, and hence for the volume of the sphere.

9 This question number is interesting because it is the largest perfect square number which has no even digits. In this question we shall prove this slightly surprising fact.

Let us start by considering n^2 for $n > 4$.

(a) If n is **even** then what can you say about n^2 ?

(b) What does your answer to (a) mean you can say about the last (units) digit of n^2 ?

Now suppose n is **odd** instead, and since $n > 4$ (and we can quickly see that 5, 7, 9 do not work) let us write $n = 10a+b$ where b is the **odd** units digit of n. The other parts of n will be $10\times$(something) – this is the $10a$, where a is a whole number.

(c) Show that $n^2 = 100a^2 + 20ab + b^2$

(d) What are the last two digits of $100a^2$?

(e) What are the possible values of b^2 ? And what about the tens digit (if there is one)?

(f) What is the units digit of $20ab$?

(g) Explain why the tens digit of $20ab$ will be the last digit of $2ab$.

(h) Is this last digit even or odd?

All this means we have an odd units digit (no carries) for n^2. Furthermore,

(i) What then can we definitely **always** say about the tens digit of n^2 (n is still odd) ?

(j) What can we conclude (combining anything from (a) to (i)) regarding perfect square numbers with only odd digits?

PTO for last question

10 The headmaster creates a scholar's garden at King's in the shape of a (5,12,13) right-angled triangle.

In the right-angled corner he installs a square shed, and from this shed the minimum distance to the hypotenuse of the triangle is 2 m. The rest of the garden is planted with grass.

What fraction of the garden area is grass?

END OF QUESTION PAPER