The King's School Canterbury

13+ Scholarship Examinations 2019

MATHEMATICS II

1 hour (including five minutes suggested reading time)

Use the reading time wisely; gain an overview of the paper and start to think of how you will answer the questions.

Do as many questions as you can (clearly numbered) on the lined paper provided. Clearly name each sheet used.

*The questions are not of equal length or mark allocation. Move on quickly if stuck. Y****ou are not expected to finish everything.***

You are expected to use a calculator where appropriate, but you must show ***full and clear working****, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments.

This paper has eight questions.

1 How many whole numbers between 1 and 2019 inclusive are **not** divisible by 3 or 7?

2 Theresa and Boris play a game in which there are only winners and losers i.e. no draws.

Each of them is equally likely to win each game.

The first player to win four games becomes champion and then the game stops.

Theresa wins the first two games. What is the probability that Boris ends up as champion?

3 Lamech is as old as Methuselah was when Lamech was as old as Methuselah had been when Lamech was half as old as Methuselah is.

Their present ages sum to 1650.

How old is Methuselah now?

4 I am marking twelve scholarship papers and decide to give them grades at random, with an equal number of grades A, B, C and Z.

(a) Explain carefully why there are 220 different ways of allocating the A grades.

(b) Work out the total number of ways of assigning the grades A, B , C, and Z as described.

5 In this question you are given that a Pythagorean triple is a set of three whole numbers *a,b,c* such that

$$a^2 + b^2 = c^2$$

(a) Explain why an odd number may be written in general as $2n - 1$.

(b) Professor Atiyah says: "If you square an odd number, then halve it and use the whole numbers either side of the answer you have a Pythagorean triple".

Use algebra to prove carefully that this statement is true.

[e.g. $9^2 = 81$, $81/2 = 40.5$, and then (9,40,41) is a Pythagorean triple, with $9^2 + 40^2 = 41^2$.]

6 In this question you can use the following definitions:

If A and B are two numbers, their three main means are

Arithmetic	Geometric	Harmonic
$\dfrac{A+B}{2}$	$\sqrt{AB}$	$\dfrac{2}{\frac{1}{A}+\frac{1}{B}}$

You can also use the fact that, if A ≠ B then Arithmetic mean > Geometric mean > Harmonic mean .

In the First Century AD Hero of Alexandria devised a way of finding approximate values for square roots e.g. $\sqrt{3}$

Consider two numbers a and $b = 3/a$.

(a) What is their geometric mean?

(b) Write down an expression for the arithmetic mean.

(c) Write an expression for the harmonic mean and simplify as much as you can.

(d) Combine these to form an inequality.

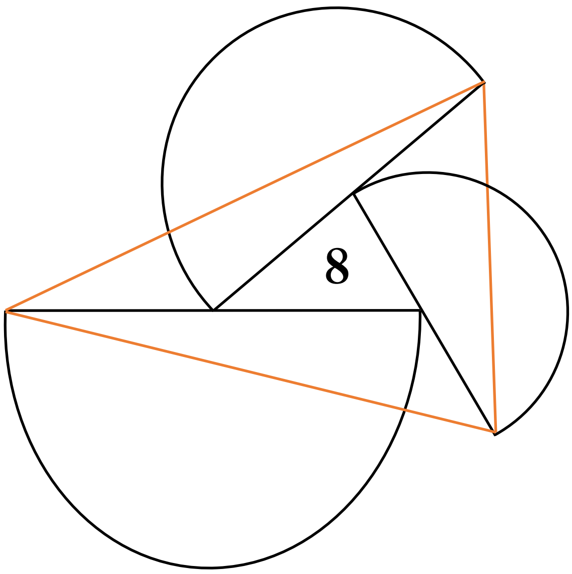
(e) Take a starting value of $a = 5/3$, and using the inequalities derived above, show that

$$\frac{45}{26} < \sqrt{3} < \frac{26}{15}$$

and then taking $a = \frac{26}{15}$ show that

$$\frac{2340}{1351} < \sqrt{3} < \frac{1351}{780}$$

7 In the diagram the vertices of the smaller triangle (of area 8cm^2) are also the centres of the semicircles. What is the area of the larger triangle?



Not to scale

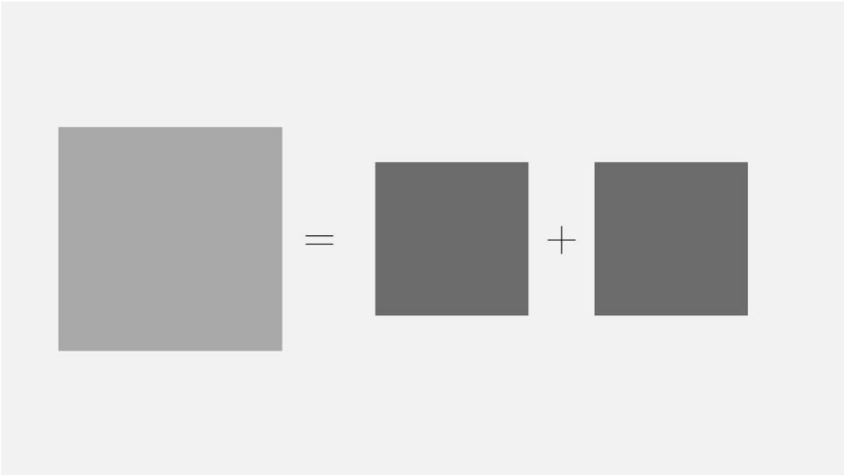
8 In this question we will investigate what sort of number $\sqrt{2}$ is.

Suppose $\sqrt{2}$ is a fraction, so we can write $\sqrt{2} = p/q$ for some **whole numbers** p and q.

(a) Rearrange this to write p^2 in terms of q^2 .

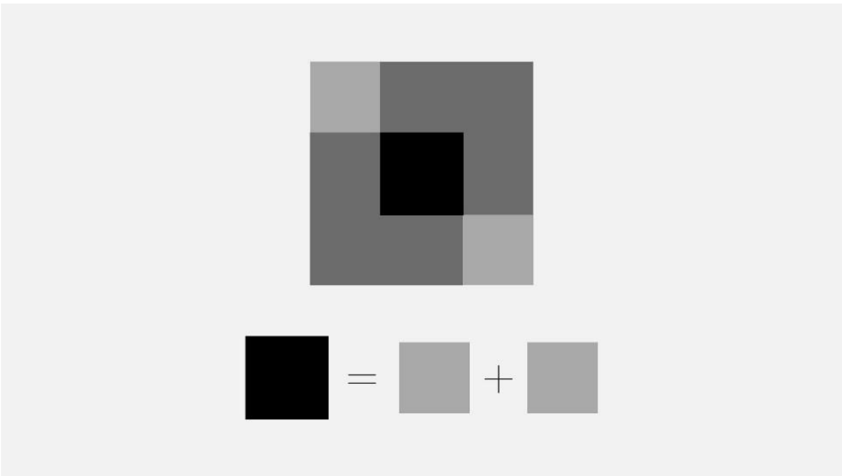
(b) If this fraction p/q is in **lowest terms** what can we say about p and q? Explain also why we can always cancel a fraction down to lowest terms.

(c) What is the connection between the picture of square areas below and parts (a) and (b)? What type of numbers are the side lengths of these squares?



(d) Placing the two smaller squares on the larger one, what must the sum of the areas of the corner squares equal?

(e) Why?



(f) Why are these smaller squares' side lengths all whole numbers?

(g) Now try to consider your answers to (d) to (f) with your answers to (a) to (c). What is the problem?

(h) What can you conclude?

END OF QUESTION PAPER

The King's School Canterbury

13+ Scholarship Examinations 2018

MATHEMATICS II

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*The questions are not of equal length or mark allocation. Move on quickly if stuck. Y**ou are not expected to finish everything.***

*You are expected to use a calculator where appropriate, but you must show **full and clear working**, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments.

This paper has ten questions. You can attempt them in any order.

1 In another scholarship examination, candidates sit six papers in total, in six time slots all on one day! Of these, they must take three different mathematics papers: number, algebra and shape.

If the rule is that we are not allowed to sit two mathematics papers in **consecutive** slots, in how many different ways can we arrange the mathematics papers? Note that we are not interested in what is being taken in the other slots, but (say) arrangement N*A*S* is different to N*A**S.

2 This question is taken from the University of London Elementary Mathematics examination in 1940.

A garrison of 1075 men could exist on full rations for 30 days. After 16 days on full rations the garrison is augmented by 129 men and, at the same time, the stock of the existing provisions is increased by 60% by means of parachute supplies. How much longer can the augmented garrison hold out on *half* rations?

See if you can answer the question, showing all your working. [Vocab: augmented means added to.]

3 Post Brexit (the UK leaving the European Union) it is decided that the country will revert to the pre-decimal currency system, of **pounds**, **shillings** and **pence** (abbreviated £ s d), which ran from before the Norman Conquest until 1971.

There were twelve pence in a shilling and twenty shillings in a pound. The half-penny is also reintroduced.

Mr Stuart buys a Royal Wedding commemorative plate and later finds that its value has gone up by 35%.

It is now worth £69 18s 7 ½ d.

What did he pay for it originally? Give your answer in £ s d.

4 Four players play a two-player dice game where they each have their own, non-standard six-sided die. The two players roll their dice and the higher score wins.

The dice are numbered like this:

A	1	1	5	5	5	5

B	4	4	4	4	4	4

C	3	3	3	3	7	7

D	2	2	2	6	6	6

Here is a table for A versus B, showing who wins in each cell

AvB	4	4	4	4	4	4
1	B	B	B	B	B	B
1	B	B	B	B	B	B
5	A	A	A	A	A	A
5	A	A	A	A	A	A
5	A	A	A	A	A	A
5	A	A	A	A	A	A

(a) Clearly A wins more often; what is the probability that A wins each game against B?

(b) Without spending time drawing grid-lines etc., quickly draw similar tables for B versus C, C versus D, and D versus A. In each case describe which player's die is stronger and what the probability is of winning for the stronger player.

(c) Write some brief comments about which is the strongest die here, or anything else relevant you wish to mention.

5 Pansy has been given a new game called Dobble (also known as Spot-It!)

There are 57 game cards each containing eight pictures of various symbols. There are 57 different symbols within the full set of cards.

One way to play the game is like an extended version of "Snap". Players turn over two cards and then everyone must try to be first to spot the **unique** matching symbol on each pair of cards (and shout it out). Note: the cards are designed so that there is always **exactly one** common symbol between any pair in the pack.

The winning player each time then keeps those cards and the person with the most cards at the end is the final winner.

(a) Why must each symbol appear eight times in total in the pack of cards?

Pansy looks at the 57 cards and removes the eight cards which contain a dinosaur symbol.

(b) Explain carefully why all 57 different symbols appear on these eight cards.

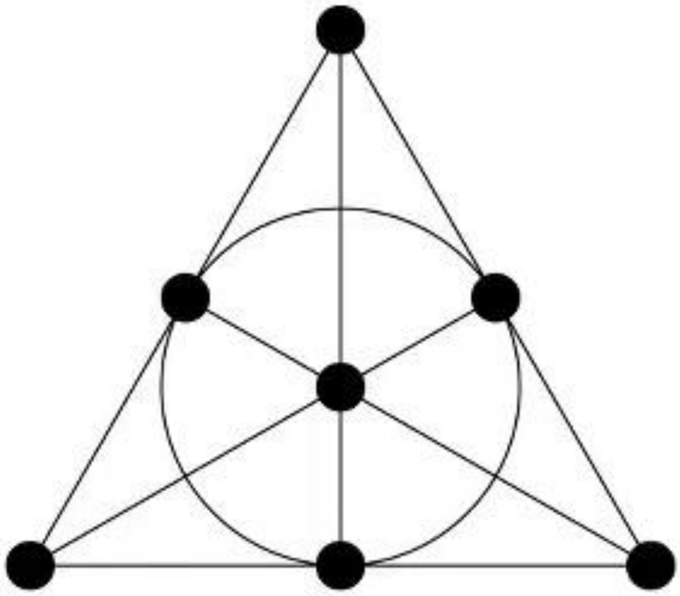
Trying to understand it better she now considers a simpler version of the game, this time with **seven** cards, and **seven** different symbols, with **three** symbols on each card. Once again, any pair of cards must have only **one** matching symbol.

The symbols on the cards are: Anchor, Bomb, Cheese, Dinosaur, Exclamation Mark, Fish, Ghost (abbreviated A, B, C, D, E, F, G).

(c) How many times will each symbol appear in this pack of cards?

(d) Write out the complete set of seven cards (use the abbreviations).

(e) What is the connection between this set of cards and the diagram opposite? There are seven lines (one of them being a circle), and there are seven points. Note that each line passes through three points, and any two points only have one line between them.



6 In an extraordinary development, someone has cheated during this examination. After the paper, Mr Miles questions the four suspects.

Walter says: Xavier cheated.
Xavier says: Walter is lying.
Yannick says: Walter cheated.
Zebedee says: I did not cheat!

(a) Suppose exactly one of these statements is true. Who cheated?

(b) Suppose exactly one of these statements is false. Who cheated?

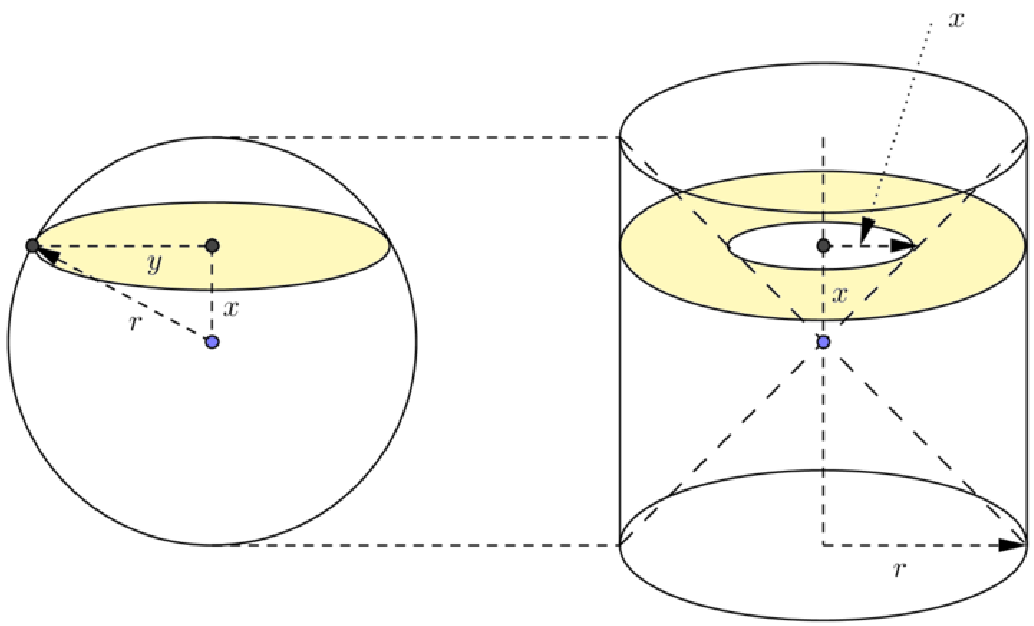
7 Mr Felton and Mr Mawby live at each end of a road. They each run from their own house to the other's and back. They each run at their own constant speed. They pass each other the first time a distance x from Mr Felton's house, then reach each other's house and immediately return to their own. On the way back they meet a second time a distance y from Mr Mawby's house.

Find an expression for the distance between their houses in terms of x and y.

8 In this question we will use Cavalieri's principle, which states that if corresponding cross-sectional areas of two figures are equal for every such "slice", then the two figures must have the same volume.

We seek a formula for the volume of a sphere, and we will assume two other volume formulae to be known already:

- For a cylinder of radius r and height h we have $V = \pi r^2 h$
- For a cone of radius r and height h we have $V = \frac{1}{3}\pi r^2 h$



In the diagrams we have a sphere on the left and, on the right, a cylinder with two cones cut out (which meet in the point shown in the very centre of the cylinder). Note that the sphere of diameter $2r$ would fit perfectly inside the cylinder of height $2r$.

(a) Explain why the shaded circle in the sphere has area $A = \pi(r^2 - x^2)$.

(b) Explain why the shaded annulus (ring) in the partly hollowed-out cylinder has area $A = \pi r^2 - \pi x^2$.

(c) What does this mean we can say about the two shaded areas?

(d) What can we deduce from this using Cavalieri's principle above?

(e) Explain why the volume of the cylinder is $V = 2\pi r^3$.

(f) Explain why the volume of the two cones is $V = \frac{2}{3}\pi r^3$.

(g) Use (e) and (f) to find the formula for the volume of the partly hollowed-out cylinder, and hence for the volume of the sphere.

9 This question number is interesting because it is the largest perfect square number which has no even digits. In this question we shall prove this slightly surprising fact.

Let us start by considering n^2 for $n > 4$.

(a) If n is **even** then what can you say about n^2 ?

(b) What does your answer to (a) mean you can say about the last (units) digit of n^2 ?

Now suppose n is **odd** instead, and since $n > 4$ (and we can quickly see that 5, 7, 9 do not work) let us write $n = 10a+b$ where b is the **odd** units digit of n. The other parts of n will be $10\times$(something) – this is the $10a$, where a is a whole number.

(c) Show that $n^2 = 100a^2 + 20ab + b^2$

(d) What are the last two digits of $100a^2$?

(e) What are the possible values of b^2 ? And what about the tens digit (if there is one)?

(f) What is the units digit of $20ab$?

(g) Explain why the tens digit of $20ab$ will be the last digit of $2ab$.

(h) Is this last digit even or odd?

All this means we have an odd units digit (no carries) for n^2. Furthermore,

(i) What then can we definitely **always** say about the tens digit of n^2 (n is still odd) ?

(j) What can we conclude (combining anything from (a) to (i)) regarding perfect square numbers with only odd digits?

PTO for last question

10 The headmaster creates a scholar's garden at King's in the shape of a (5,12,13) right-angled triangle.

In the right-angled corner he installs a square shed, and from this shed the minimum distance to the hypotenuse of the triangle is 2 m. The rest of the garden is planted with grass.

What fraction of the garden area is grass?

END OF QUESTION PAPER

13+ Scholarship Examinations 2017

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This paper has seven questions.

1 Every pupil at King's is either a scholar or not, and (for this question) is either a girl or a boy.

Girl scholars always tell the truth and girl non-scholars always lie.
Boys are the other way around i.e. scholars always lie, non-scholars always tell the truth.

The Girl says: I am a scholar.

The Boy says: that is true.

What are they each? Is there more than one answer? Show your reasoning carefully.

2 Baldrick has twice as many turnips as he has carrots.

He eats ten pieces of each kind, and now has three times as many turnips as carrots.

How many turnips did he originally have?

3 Boris used to sell spinach to the UK and Europe **only**, with 99% of it going to Europe.

Even post-Brexit he still sells 95% to Europe, while selling exactly the same amount in the UK as before; he does not sell any to anyone else.

Explain carefully why this is a worrying development, showing any calculations.

4 Suppose that x and y are non-zero numbers such that

$$\frac{5x+y}{x-5y}=3$$

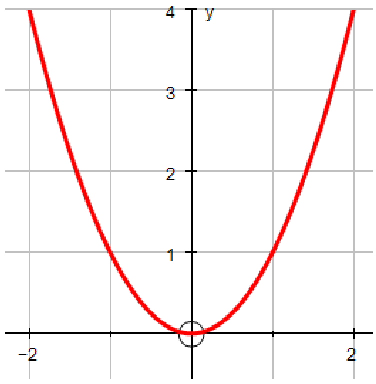
What is the value of

$$\frac{x+5y}{5x-y}$$

?

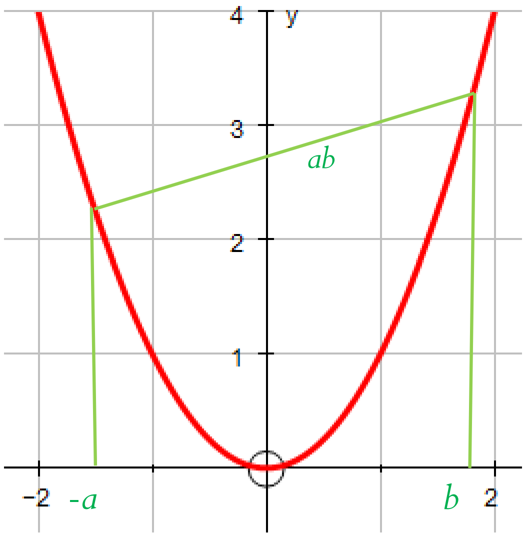
5 Professor Möbius draws a graph of $y = x^2$

(here shown from x values from -2 to 2).



Möbius claims that he can use this graph as a new way of multiplying two numbers together.

He says: "If I want to multiply a and b, I find the points on the graph at $x = -a$ and $x = b$. I then join these with a straight line, and where this crosses the y-axis will be the answer, ab."



Show carefully using algebra that Professor Möbius is correct. (Copy the diagram into your script as needed.)

[In this question you might find the following algebraic fact (the "difference of two squares") useful:

$$a^2 - b^2 = (a + b)(a - b)$$

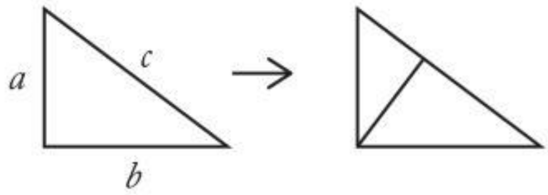
.]

6 In this question we are going to prove the Pythagorean Theorem, namely

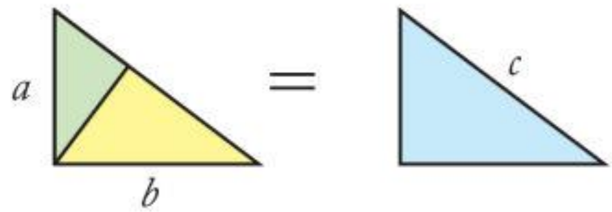
$$a^2 + b^2 = c^2$$

for a right-angled triangle of sides a, b, c (c is the **hypotenuse**). The proof you might reconstruct below is sometimes attributed to a twelve-year-old Albert Einstein.

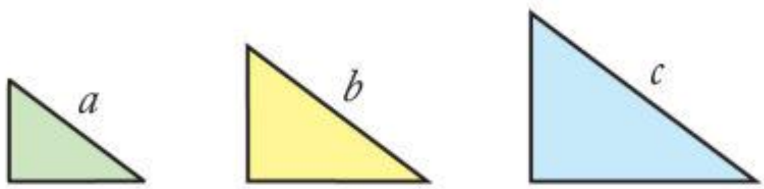
Draw a perpendicular line from the hypotenuse to the right angle, which divides the original right-angled triangle into two smaller right-angled triangles.



(a) What is the connection between the little triangle area, the area of the medium triangle and that of the original triangle?

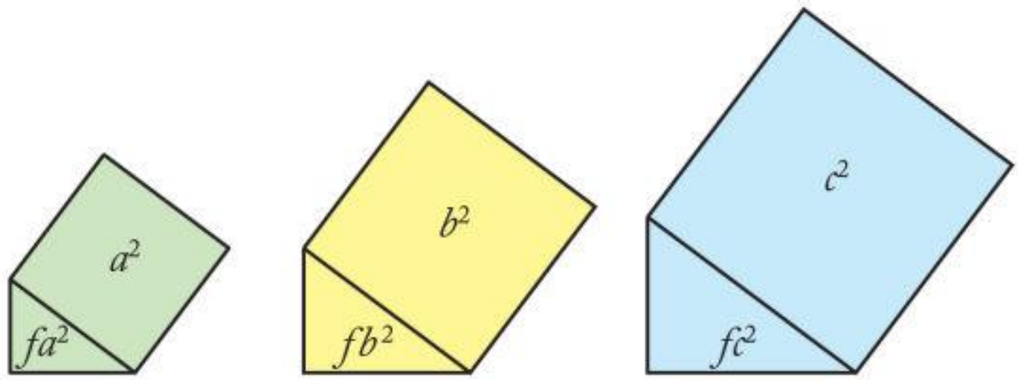


(b) Explain why the three triangles are in fact **similar** (same shape, not necessarily same size).



(c) Explain the following: because the triangles are similar, each will occupy the same fraction f of the area of the square on its hypotenuse.

(Restated symbolically, this observation says that the triangles have areas fa^2, fb^2, and fc^2, as indicated in the diagram.)



(d) Explain why $fa^2 + fb^2 = fc^2$.

(e) How does this give us the Pythagorean theorem?

7 On the pages overleaf is a type of poem called a **Sestina** written in a mediaeval language (Old Occitan) by Arnaut Daniel in about 1200.

Do not try to read the poem; we are only interested in the pattern of the last words on each line.

The main part of the poem consists of six stanzas (verses) where the last words on each line cycle according to a fixed rule. (For now, we ignore the "envoi" (the last three lines)).

The numbering (summarised again below) shows how the end-words cycle around from stanza to stanza.

I	II	III	IV	V	VI
1	6	3	5	4	2
2	1	6	3	5	4
3	5	4	2	1	6
4	2	1	6	3	5
5	4	2	1	6	3
6	3	5	4	2	1

(a) If the rearrangement continued, what would be the ordering for the seventh stanza?

(b) What might be the simplest rule that left nothing unchanged each time, but then returned to the starting arrangement after six stanzas?

(c) What happens if I try writing a sestina using the following rule for rearrangement: 123456 in the first stanza moves to 312645 in the second (and then repeating)?

Use the following information in the next two parts.

There are 720 = 6! possible rearrangements of six words.
There are 265 of these possible rearrangements where nothing ends up in the same position.
Of these, 120 will take six repeats to return to the starting point.
90 will take four repeats to return to the starting point.
40 will take three repeats to return to the starting point.
15 will take two repeats to return to the starting point.

(d) Suppose Arnaut Daniel chooses the rearrangement rule at random. What is the probability that he chooses a rearrangement where at least one word ends up in the same position as before?

(e) Another writer chooses at random and she picks one where nothing ends up in the same position. What is the probability that it will return to the starting point in less than six repeats?

Another way of generating last words in a poem follows this rule (with *n* taking number values 1 to 6)

Input *n*
Output:

- If *n* is even, write $\frac{n}{2}$ in that position
- If *n* is odd, write $\left(6 - \frac{n-1}{2}\right)$ in that position.

(f) Starting with a first stanza 123456 as before write down a sequence of numbers to show how the next five stanzas' lines will end using this rule. Comment on your answers.

Lo ferm voler q'el cor m'**intra**	1	Stanza I
no'm pot ies becs escoissendre ni **ongla**	2	
de lausengier, qui pert per mal dir s'**arma**	3	
e car non l'aus batr'ab ram ni ab **verga**	4	
si vals a frau lai o non aurai **oncle**	5	
jauzirai joi, en vergier o dinz **cambra**	6	
Qan mi soven de la **cambra**	6	Stanza II
on a mon dan sai que nuills hom non **intra**	1	
anz me son tuich plus que fraire ni **oncle**	5	
non ai membre no'm fremisca, neis l'**ongla**	2	
aissi cum fai l'enfas denant la **verga**	4	
tal paor ai no'l sia trop de l'**arma**	3	
Del core li fos non de l'**arma**	3	Stanza III
e cossentis m'a celat deniz sa **cambra**	6	
que plus mi nafra'l cor que colps de **verga**	4	
car lo sieus sers lai on ill es non **intra**	1	
totz temps serai ab lieis cum carns et **ongla**	2	
e non creirai chastic d'amic ni d'**oncle**	5	
Anc la seror de mon **oncle**	5	Stanza IV
non amei plus ni tant per aqest'**arma**	3	
c'aitant vezis cum es lo detz de l'**ongla**	2	
s'a liei plagues volgr'esser de sa **cambra**	6	
de mi pot far l'amors q'inz el cor m'**intra**	1	
mieills a son vol c'om fortz de frevol **verga**	4	
Pois flori la seca **verga**	4	Stanza V
Ni d'en Adam mogron nebot ni **oncle**	5	
tant fin'amors cum cella q'el cor m'**intra**	1	
non cuig fos anc en cors ni eis en **arma**	3	
on q'ill estei fors on plaz' o dins **cambra**	6	
mos cors no' is part de lieis tant cum ten l'**ongla**	2	
C'aissi s'enpren e s'en**ongla**	2	Stanza VI
mos cors e lei cum l'escorss'en la **verga**	4	
q'ill m'es de joi tors e palaitz e **cambra**	6	
e non am tant fraire paren ni **oncle**	5	
q'en paradis n'aura doble joi m'**arma**	3	
si ja nuills hom per ben amar lai **intra**	1	
Arnautz tramet sa chanson d'**ongl**'e d'**oncle**	2,5	Envoi
a grat de lieis que de sa **verg'a** l'**arma**	4,3	(ignore this part)
son Desirat cuit pretz en **cambra intra**	6,1	

END OF PAPER

13+ Scholarship Examinations 2016

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The questions are not of equal length or mark allocation. Make sure you avoid spending too much time on any one question; don't get bogged down! Move on quickly if you get stuck. The paper is long; ***you are not expected to finish everything.***

Some of the later questions are more difficult, but not necessarily longer. Some questions are designed to test your ability to work with unfamiliar ideas, or familiar ones with a twist. Don't give up!

You are expected to use a calculator where appropriate, but you must show ***full and clear working****, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments. You can sometimes, however, manage to complete later parts of questions, even if you have failed to answer the earlier sections.

This paper has nine questions.

1 Vic is awarded £X prize money for scholarship performance and Bob is awarded £Y.

Vic has won more than Bob and they both win an odd number of pounds.

Find an algebraic expression for the smallest (whole) number of pounds Vic needs to give Bob so that Bob has more money than Vic.

2 There are several choirs singing at King's. Membership of choirs is linked as follows:

- All members of Crypt Choir are in Chapel Choir
- Almost all Madrigalia are in Crypt Choir
- All of Chapel Choir are in Nave Choir
- All of Becket singers are in Nave Choir.
- All Nave Choir and all Girls' Choir are in King's Chorus.
- Other groups include King's Men, King's Swingers and King's A Capella.

State whether each of the following is true or false (or you cannot determine an answer), giving brief explanations for each.

(a) I am in Crypt Choir, therefore I am in King's Chorus.

(b) I can be a Becket Singer and in Chapel Choir.

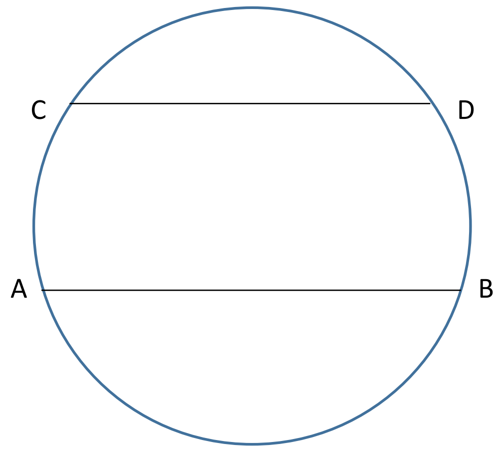
(c) I am in Crypt Choir, therefore I am in Madrigalia.

(d) I am not in the Nave Choir, therefore I am in neither Chapel nor Crypt.

(e) The Nave Choir is larger than the King's Chorus.

3 AB and CD are parallel chords (lines) on a circle of length 14cm and 10cm respectively. The chords are 6cm apart. Find the distance of the centre of the circle from the line AB.

NOT TO SCALE



[Hint: you may wish to draw a diagram and add some radii to it.]

4 Suppose that W, X, Y, Z are connected by the formula

$$W = \frac{XY}{Z}$$

If X is increased by 5%, Y is decreased by 68% and Z is decreased by 46%, work out the percentage change in W, stating whether it is an increase or a decrease.

5 Another school brings in a new scholarship selection method. For each subject for which they are entered, candidates pick (at random) a ball out of a bag containing 100 balls (numbered 1,2,3,…,100), and then replace it.

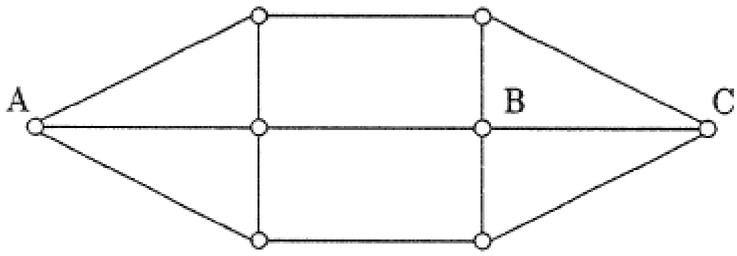
The candidate is classified as **weird** if he/she picks out ball 1 or ball 100 for any subject. Candidates are either weird or not weird.

So, we expect 2% of candidates entering only one subject to be weird.

Suppose the school has a very large number of scholarship candidates.

(a) Explain, showing working, why we expect 3.96% of candidates entering two subjects to be weird.
(b) Work out the expected percentage of weird candidates sitting three subjects.
(c) By extending any pattern in the calculations you have been doing, show that if candidates are put in for fifteen subjects then more than a quarter of them will be expected to be weird.
(d) What do you think is the pattern to these percentages as we increase the number of subjects taken. Why do you think this is so?

6 Some rats have a network of tunnels as shown. Consider a rat. Each night he sleeps at one of the junctions. Each day he moves to a neighbouring junction but chooses a path at random (each of equal likelihood from those available).



A rat starts at A.

(a) What is the probability that two nights later he is at B?

Another rat starts at B

(b) Calculate the probability that two nights later he is back at B again.

Two more rats start, one at C and the other at A

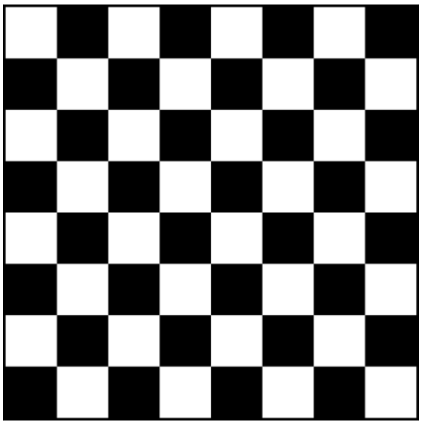
(c) Calculate the probability that two nights later they are at the same junction.

7 Alfred tosses a coin until he throws a head and then stops.

He claims that the probability that he stops on an even-numbered turn is ½ .

Explain why he is wrong and work out the actual probability that he stops on an even-numbered turn.

8 Consider a standard 8x8 chessboard.



(a) How many different ways are there of placing two identical rooks (pieces) on an 8×8 chessboard?

(b) Now suppose we have a white King and a black King, which are **not** allowed to occupy adjacent squares. In how many different ways can we place them on the chessboard?

NB the chessboard is of fixed orientation i.e. if two possible configurations of pieces differ by a rotation then we do count them as being different in both parts of this question.

9 A, B, C are three scholarship candidates, sitting the examinations, not necessarily each taking the same number of test papers.

A has a mean average score of 40 points.
B has a mean score of 50 points.
Taken together A and B have a mean score of 43 points.
Taken together A and C have a mean score of 44 points.
What is the greatest possible integer (whole number) mean score for B and C combined?

END OF PAPER